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WEAKLY NONLINEAR KELVIN–HELMHOLTZ WAVES BETWEEN FLUIDS OF FINITE DEPTH

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Abstract—Weakly nonlinear, gravity waves at the interface between two fluids in relative motion are considered. The dynamical limit to progressive waves of permanent form (an extension to finite amplitude of the Kelvin–Helmholtz instability) is studied as a function of the fluid depths. Stable, gas/liquid waves are shown to exist at current velocities above U_{cl}, the critical of linear theory (supercritical stability). A notable exception is shown to hold for a range of wavelengths when a liquid of infinite extent is bounded by a thin gas film. Long, liquid/liquid waves cease to exist at current velocities below U_{cl} and are further unstable to self-interaction with higher harmonics (subcritical instability). These notions are used to discuss the nonlinear evolution of interfacial instabilities.

Key Words: Kelvin–Helmholtz instability, finite-depth effects, nonlinear evolution

1. INTRODUCTION

Over the last few decades a large amount of work has been done on inviscid, steady progressive, gravity waves on a free surface. The neglect of air is justified in oceanographic applications on the grounds of its small density and the large wavelength scales, requiring unrealistically high wind speeds to exhibit significant air influence. The situation in horizontal, two-phase flows is quite different. Indeed, the Kelvin–Helmholtz instability, a strong manifestation of the effect of air flow, is of central importance in understanding various observations (e.g. Kordyban & Ranov 1970; Andritsos & Hanratty 1987). The extension of the analytical and numerical techniques developed for handling free-surface waves could point to useful new results for interfacial waves and, consequently, could lead to a better understanding of some two-phase flow phenomena. Therefore, it is attempted in the present work to develop on the theory of interfacial waves between two fluids in relative motion, when one or both layers are shallow with respect to the wavelength.

Nonlinear interfacial waves have been considered by relatively few investigators. An excellent review of the early work in the field is provided by Miles (1986). Pullin & Grimshaw (1983a, b) derived a third-order expansion for arbitrary depths and obtained numerical solutions for nonlinear progressive waves for a thin upper layer in the Boussinesq limit. Miles (1986) derived a second-order expansion for arbitrary depths and provided the evolution equations that govern Kelvin–Helmholtz waves in the parametric neighborhood of the critical point. Bontozoglou & Hanratty (1988; hereafter referred to as BH1), following Saffman & Yuen (1982) who considered fluids of infinite extent, examined the "dynamical limit" to the existence of steady wave solutions in the case of a thin liquid film bounded by a gas of infinite extent.

The dynamical limit is conveniently defined by considering an increasing current velocity U with a fixed wave height. For U larger than a critical velocity U_c, solutions at a given wave height cease to exist althrough the limiting wave profile is smooth and exhibits no unphysical properties. For infinitesimal heights this limit is associated with the well-known Kelvin–Helmholtz instability; it may be interpreted as the nonexistence of steady linear waves of a given wavelength when U is sufficiently large. For finite-amplitude waves the critical current U_c is a function of the wave height.

The present work examines the characteristics of long gravity waves at the interface between two shallow fluid layers (gas/liquid or liquid/liquid). The effect of surface tension is neglected in view of the large length scale of the waves considered. Relative motion of the fluids is included by considering a nonzero current velocity U. The dynamical limit is studied as a function of the flow parameters, using a perturbation expansion in the wave amplitude. In this sense the present contribution is complimentary to previous work (BH1), which examined only the effect of finite lower-fluid depth.

Attention is focused in the parametric neighborhood of the Kelvin–Helmholtz instability, where existence or nonexistence of progressive waves of permanent form is observed by varying system parameters. A fundamentally different physical picture emerges for the two cases with implications in the evolution of the interface with increasing relative velocity U. The stability of weakly nonlinear waves under self-interaction is considered and both supercritical stability and subcritical instability are predicted for different conditions. Finally, the results are discussed in connection with the stability analysis of Ahmed & Banerjee (1985) and inferences are made regarding the possible evolution of long waves.

2. PROBLEM FORMULATION

Periodic gravity waves are considered at the interface between two fluids of uniform depth. The fluids have different densities and the upper is moving relative to the lower with a horizontal velocity U. They are taken to be incompressible and inviscid, and the motion is assumed to be irrotational. With the above assumptions, the flow can be described by two velocity potentials, φ_1 and φ_2. Solutions are obtained for steady, two-dimensional waves of wavelength L, which propagate with phase speed C in the direction of U. Properties of the lower fluid are subscripted by 1 and those of the upper fluid by 2. The two fluids are assumed to be stably stratified by gravity, so $\rho_2 < \rho_1$.

The flow is sketched in figure 1. Rectangular coordinates (x, y) are chosen such that the x-axis is horizontal and the y-axis is directed vertically upwards. The interface is located at $y = \eta$ and the bottom and top boundaries at $-d_1$ and d_2, respectively. The origin is chosen so that the mean elevation $(\eta)_{\text{mean}} = 0$. The reference frame is such that the fluid velocity averaged over one wave cycle (circulation), at any fixed depth within the lower fluid, is zero. For finite upper fluid depth, the current velocity U is defined similarly. It equals the fluid velocity averaged over one cycle, on any fixed height within the upper fluid. For an unbounded upper fluid, U is simply the fluid velocity at infinity.

With the above assumptions, the velocity potentials φ_1 and φ_2 satisfy Laplace's equation in each fluid domain, in addition to the following set of boundary conditions:

(i) *the no penetration conditions, [1] and [2], on the solid boundaries,*

$$\frac{\partial \varphi_1}{\partial y} = 0 \quad \text{at } y = -d_1 \qquad [1]$$

and

$$\frac{\partial \varphi_2}{\partial y} = 0 \quad \text{at } y = d_2; \qquad [2]$$

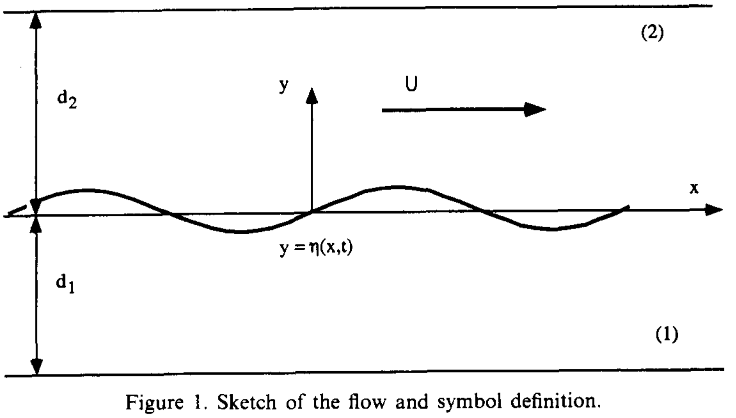

Figure 1. Sketch of the flow and symbol definition.

(ii) *the kinematic conditions at the interface,*

$$\frac{\partial \eta}{\partial t} + \frac{\partial \varphi_i}{\partial x}\frac{\partial \eta}{\partial x} = \frac{\partial \varphi_i}{\partial y} \quad \text{at } y = \eta \ (i = 1, 2); \qquad [3]$$

and

(iii) *the dynamic boundary condition which, with the neglect of surface tension, guarantees continuity of pressure across the interface,*

$$\left[\frac{\partial \varphi_1}{\partial t} + \tfrac{1}{2}(\nabla\varphi_1)^2\right] - r\left[\frac{\partial \varphi_1}{\partial t} + \tfrac{1}{2}(\nabla\varphi_2)^2\right] + (1-r)g\eta + \mathrm{K} = 0, \qquad [4]$$

where

$$r = \frac{\rho_2}{\rho_1}$$

and

$$\mathrm{K} = \text{combination of Bernoulli constants.}$$

3. WEAKLY NONLINEAR APPROXIMATION

The properties of weakly nonlinear steady waves are obtained by using Whitham's averaged variational principle (Whitham 1974, Section 16.6). For the case of interest the averaged Lagrangian is given by

$$L = -\overline{\int_{-d_1}^{\eta}\left[\frac{\partial \varphi_1}{\partial t} + \tfrac{1}{2}(\nabla\varphi_1)^2 + gy\right]\mathrm{d}y} + \overline{\int_{d_2}^{\eta}\left[r\frac{\partial \varphi_1}{\partial t} + \tfrac{1}{2}r(\nabla\varphi_2)^2 + rgy\right]\mathrm{d}y} - L_0, \qquad [5]$$

where the overbar denotes averaging over one cycle of the wave phase and

$$L_0 = -\int_{-d_1}^{\eta} gy\,\mathrm{d}y - \int_0^{d_2}(\tfrac{1}{2}rU^2 + rgy)\,\mathrm{d}y; \qquad [6]$$

L_0 is included only when one or both boundaries move to infinity, in order to ensure a convergent value for L. Following Whitham (1974), the leading-order terms for the wave profile and the velocity potentials are substituted in the expression for L:

$$\begin{aligned}
\eta(w) &= a\cos w + a_2\cos 2w,\\
\varphi_1(x,y,t) &= A_1[\exp(ky) + \exp(-2kd_1)\exp(-ky)]\sin w\\
&\quad + \frac{A_2}{2}[\exp(2ky) + \exp(-4kd_1)\exp(-2ky)]\sin 2w,\\
\varphi_2(x,y,t) &= Ux + B_1[\exp(ky) + \exp(2kd_2)\exp(-ky)]\sin w\\
&\quad + \frac{B_2}{2}[\exp(2ky) + \exp(4kd_2)\exp(-2ky)]\sin 2w,
\end{aligned} \qquad [7]$$

where w is the wave phase $w = kx - \omega t$ and, for progressive waves of permanent form, the coefficients a, A_1 and B_1 are real.

It is anticipated that A_1 and B_1 are $O(a)$ and a_2, A_2 and B_2 are $O(a^2)$, and terms up to $O(a^4)$ are retained in the expression for L. Note that the $O(a^3)$ and $O(a^4)$ terms in the expansion [7] automatically disappear during the averaging procedure, so the expression for the Lagrangian L is on the whole of accuracy $O(a^4)$. The coefficients A_1, B_1, A_2 and B_2 are eliminated by use of the equations

$$\frac{\partial L}{\partial A_1} = \frac{\partial L}{\partial B_1} = \frac{\partial L}{\partial A_2} = \frac{\partial L}{\partial B_2} = 0, \qquad [8]$$

and after some algebra it is found that

$$L = -\tfrac{1}{2}rU^2d_2 + \tfrac{1}{2}g(d_1^2 - rd_2^2) + \tfrac{1}{4}g(r-1)(a^2 + a_2^2) + \tfrac{1}{4}ka^2\left(\lambda^2\frac{1+x}{1-x} + r\lambda'^2\frac{1+y}{1-y}\right)$$
$$-\tfrac{1}{4}k^2a^2a_2\left[\lambda^2\frac{1+4x+x^2}{(1-x)^2} - r\lambda'^2\frac{1+4y+y^2}{(1-y)^2}\right] + \tfrac{1}{2}ka_2^2\left(\lambda^2\frac{1+x^2}{1-x^2} + r\lambda'^2\frac{1+y^2}{1-y^2}\right)$$
$$+\tfrac{1}{4}k^3a^4\left\{\lambda^2\left(\frac{1+x}{1-x}\right)\left[\frac{x}{(1-x)^2} - \frac{1}{4}\right] + r\lambda'^2\left(\frac{1+y}{1-y}\right)\left[\frac{y}{(1-y)^2} - \frac{1}{4}\right]\right\}, \qquad [9]$$

where $x = \exp(-2kd_1)$, $y = \exp(-2kd_2)$, $\lambda = C_1$ = linear phase speed, $\lambda' = U - C_1$ and λ and λ' are related by the linear dispersion relation

$$\lambda^2\frac{1+x}{1-x} + r\lambda'^2\frac{1+y}{1-y} = \frac{g}{k}(1-r). \qquad [10]$$

The value of a_2 is found from $\partial L/\partial a_2 = 0$ and is substituted in [9]. The dispersion relation for the weakly nonlinear wave then follows from $\partial L/\partial(a^2) = 0$:

$$\frac{1+x}{1-x}C^2 + r\frac{1+y}{1-y}(U-C)^2$$
$$= \frac{g}{k}(1-r)\left[\!\left[\, 1 + \tfrac{1}{2}k^2a^2 - 2k^2a^2\frac{y}{(1-y)^2} + 2k^3a^2\frac{\lambda^2}{g(1-r)}\left(\frac{1+x}{1-x}\right)\left[\frac{y}{(1-y)^2} - \frac{x}{(1-x)^2}\right]\right.\right.$$
$$\left.\left. + \tfrac{1}{2}k^2a^2\frac{\left\{\frac{k\lambda^2}{g(1-r)}\left[\frac{1+4x+x^2}{(1-x)^2} + \left(\frac{1+x}{1-x}\right)\left(\frac{1+4y+y^2}{1-y^2}\right)\right] - \left(\frac{1+4y+y^2}{1-y^2}\right)\right\}^2}{\left\{\left(\frac{1-y}{1+y}\right)^2 + \frac{2k\lambda^2}{g(1-r)}\left[\frac{1+x^2}{1-x^2} - \frac{1+x}{1-x}\frac{1+y^2}{(1+y)^2}\right]\right\}}\,\right]\!\right]$$
$$= \frac{g}{k}(1-r)(1 + k^2a^2S), \qquad [11]$$

where S represents the remaining terms in the large open square brackets. For fluids of infinite extent $d_1 \to +\infty$, $d_2 \to +\infty$ and $x = y = 0$. It can be verified that in this case [11] agrees with that given by Saffman & Yuen (1982).

The behavior of slowly varying solutions in the neighborhood of the Kelvin–Helmholtz instability is considered next by taking the coefficients in the expansions [7] to be functions of the "slow" time $\tau = \epsilon t$, where ϵ is a scaling parameter of the order of the wave amplitude. In other words, it is postulated that the coefficients do not change significantly over timescales comparable to the wave period. The new expansions are substituted in [5] and it is observed that, to the order of accuracy considered, derivatives of only the first-order coefficients appear in the expression. Therefore, the higher coefficients are readily eliminated by the same procedure as above and after some algebra it is found that

$$L = k^2\left\{\dot{a}^2 - \frac{\left[\frac{(1-r)g}{k} - A\right]}{T}k^2a^2 - \tfrac{1}{2}S\frac{\frac{(1-r)g}{k}}{T}k^4a^4\right\} + \text{const}, \qquad [12]$$

where

const = the constant terms, which are the same as in [9],

$$A = \lambda^2\frac{1+x}{1-x} + r\lambda'^2\frac{1+y}{1-y},$$

$$T = \frac{1+x}{1-x} + r\frac{1+y}{1-y}$$

and

$$\dot{a} = \frac{\mathrm{d}a}{\mathrm{d}t}.$$

Since the analysis is valid in the neighborhood of the Kelvin–Helmholtz instability, the phase velocity is taken equal to the neutral stability value in the linear approximation. It should be noted that [12] correctly reduces to [9] when the amplitude, a, of the wave is taken as constant ($\mathrm{d}a/\mathrm{d}t = 0$). The evolution of slowly varying waves is described by the Euler equation of the averaged Lagrangian [12]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{a}}\right) - \frac{\partial L}{\partial a} = 0, \qquad [13]$$

which is found equal to

$$k\ddot{a} + \frac{\left[\dfrac{(1-r)g}{k} - A\right]}{T} k^2(ka) + S\frac{\dfrac{(1-r)g}{k}}{T} k^2(ka)^3 = 0. \qquad [14]$$

This result will be used to examine the stability to self-interaction of the progressive solutions of permanent form.

4. RESULTS

4.1. Progressive waves of permanent form

It can be seen from the dispersion relation [11] that, for linear waves ($a \to 0$) and given values of the density ratio r, current velocity U and fluids depth d_1 and d_2, there are two solutions corresponding to the two roots of the quadratic equation for C. These are denoted by C_+ and C_-, where $C_+ > C_-$. For the linear case steady solutions cease to exist when U exceeds a critical value U_{cl} (the second subscript, l, standing for linear), given by

$$U_{cl} = \sqrt{\frac{g}{k}\left(\frac{1-r}{r}\right)\frac{\left(\dfrac{1+x}{1-x} + r\dfrac{1+y}{1-y}\right)}{\dfrac{1+x}{1-x}\dfrac{1+y}{1-y}}} \qquad [15]$$

and

$$C_+ = C_- = \frac{r\left(\dfrac{1+y}{1-y}\right)}{\left(\dfrac{1+x}{1-x} + r\dfrac{1+y}{1-y}\right)} U_{cl}. \qquad [16]$$

For finite-amplitude waves ($a \neq 0$) these two solutions continue into two families of solutions $C_+(a)$ and $C_-(a)$. From the form of the dispersion relation [11] for finite-amplitude waves it can be seen that there will again be a critical current U_c beyond which steady solutions no longer exist. Saffman & Yuen (1982) calculated U_c for unbounded fluids, both analytically (second-order approximation) and numerically. They were the first to note that the critical current velocity increases for increasing wave amplitude a, a result that can be viewed as a stabilization of parallel flows by waves. Thus, for a given value of $U > U_{cl}$, steady interfacial configurations exist on unbounded fluids only if there are waves with heights greater than some minimum.

The main focus of the present work is to examine the dependence of this phenomenon on the depth of the fluid layers. In particular, the cases of a shallow upper/deep lower layer and of shallow layers of comparable depth are considered in detail. The motivation is to provide an understanding of interfacial waves for the flow of gas and liquid in pipelines and channels, under conditions corresponding to the initiation of slugging.

The following value of the critical current velocity U_c, correct to second order in the amplitude a, is obtained by equating the two roots in [11]:

$$U_c = U_{cl}[1 + k^2a^2S]^{1/2}. \qquad [17]$$

Therefore, for the general case the term $[(U_c/U_{cl})^2 - 1]$ varies linearly with k^2a^2, with the slope S being a function of the ratio of densities, the wavelength and the fluid depths. It is demonstrated elsewhere (Bontozoglou & Hanratty 1990) that this dependence of U_c on the amplitude, a, changes when the wavelengths considered are small enough that surface tension can no longer be neglected.

The slope S is a rather complicated function of the system variables, as is evident from [11]. Therefore, it is deemed illuminating to plot a few representative cases. Fluid layers of comparable depth are considered first and results for the slope S as a function of the dimensionless depth, kd_1, are shown in figures 2a–d for $d_2/d_1 = 1$, 0.5 and 0.25. For the flow of air and water (figure 2a) and equal depths, the slope S retains an almost constant positive value over a wide range of wavelengths, then increases abruptly for very long waves. This increase occurs earlier as the gas spacing decreases. Compared with the smaller values of S for unbounded fluids, these results indicate that long, interfacial waves of very small amplitude can have a strong stabilizing effect in such gas/liquid flows. Figure 2b shows results for a ratio of densities $r = 0.1$, which could correspond to a gas/liquid system under high pressure. The similarity with figure 2a indicates that the above conclusions should also hold for high pressure. For reasons that will become clearer in the next section, this behavior will be called supercritical.

The calculations for $r = 0.9$, a density ratio typical of liquid/liquid systems, are presented in figure 2c. A striking difference with previous results is evident. The slope S decreases monotonically with increasing wavelength and becomes negative for very long waves. It is interesting to note that a negative value of S corresponds to a decrease in the critical current velocity U_c with increasing wave amplitude. Consequently, high, steady waves cease to exist at current velocities lower than the critical linear, U_{cl}, and no waves of a given wavelength exist for $U > U_{cl}$.

In previous work (BH1) it was demonstrated that this last behavior, which is now called subcritical, always occurs for a thin enough lower fluid in contact with an unbounded upper fluid. The respective curve for the dependence of S on kd_1 is included in figure 2c for comparison. The present results indicate that the effect of an upper boundary in close proximity to the interface is inverse for fluids with similar and with very different densities. In the liquid/liquid system, a shallow upper fluid is further destabilizing long interfacial waves ($U_c < U_{cl}$ for $a > 0$), an effect which, as figure 2c demonstrates, extends to shorter waves with decreasing d_2. In contrast to the above, the inclusion of an upper boundary in a gas/liquid system turns the behavior from subcritical to supercritical, as a comparison between the results in BH1 and figure 2a readily indicates.

The effect of the ratio of densities, r, on the slope S is systematically presented in figure 2d. Results are plotted for $kd_1 = 1$ and $d_2/d_1 = 1$, 0.5 and 0.25 and a transition from supercritical to subcritical behavior is evident with increasing values of r. The practical significance of the results

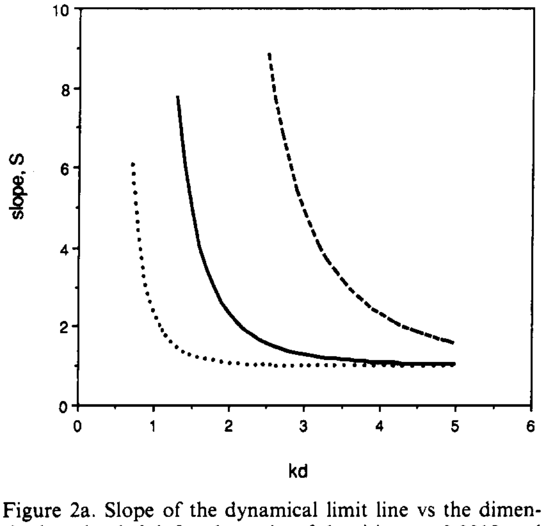

Figure 2a. Slope of the dynamical limit line vs the dimensionless depth kd_1 for the ratio of densities $r = 0.0012$ and $d_2/d_1 = 1.0$ (· · ·), 0.5 (——) and 0.25 (– – –).

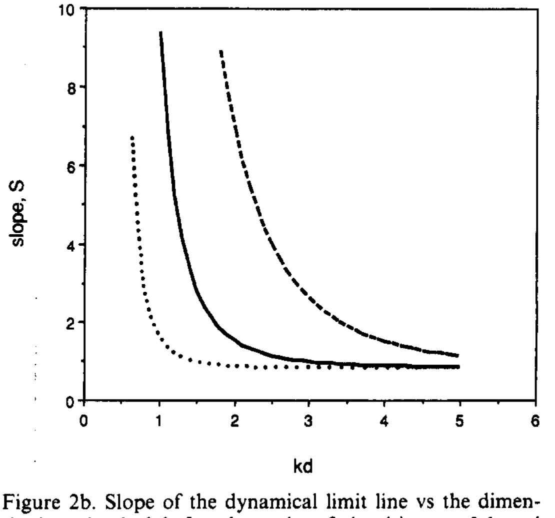

Figure 2b. Slope of the dynamical limit line vs the dimensionless depth kd_1 for the ratio of densities $r = 0.1$ and $d_2/d_1 = 1.0$ (· · ·), 0.5 (——) and 0.25 (– – –).

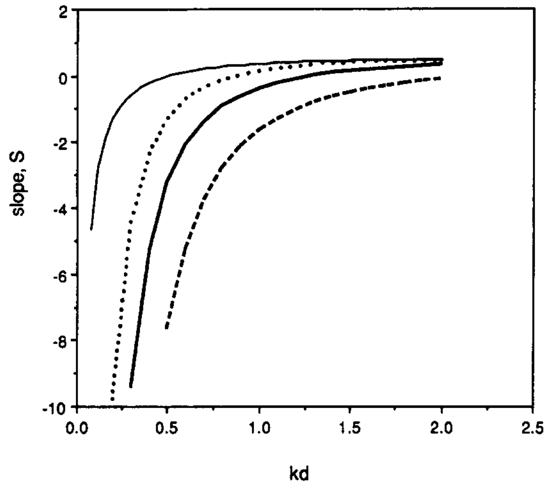

Figure 2c. Slope of the dynamical limit line vs the dimensionless depth kd_1 for the ratio of densities $r = 0.9$ and $d_2/d_1 = 1.0$ (···), 0.5 (——) and 0.25 (– – –). The thin, continuous line, corresponding to an upper fluid of infinite extent, is included for comparison.

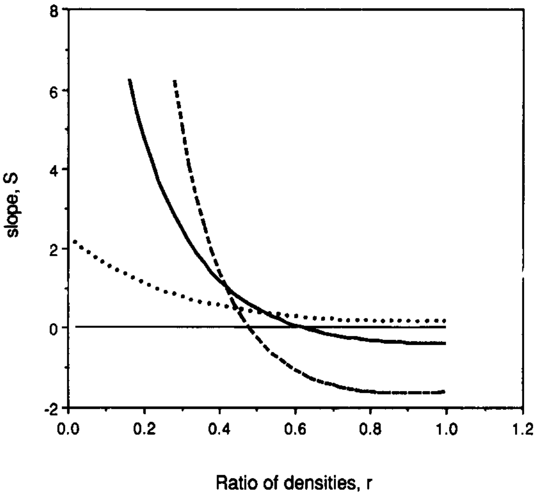

Figure 2d. Slope of the dynamical limit line vs the ratio of densities, r, for dimensionless depth $kd_1 = 1.0$ and $d_2/d_1 = 1.0$ (···), 0.5 (——) and 0.25 (– – –).

for intermediate values of the ratio of densities is, however, questionable since one can hardly find pairs of fluids with appropriate densities.

A physical explanation of the different behavior for small and large density ratios can be provided on the following lines. The Kelvin–Helmholtz instability for gravity waves is associated with a balance between the destabilizing inertia of the upper fluid (suction) and the stabilizing effect of the density stratification. The inclusion of weak nonlinearity adds $O(a^2)$ corrections to the inertia and gravity terms in both fluids. In particular, the suction exercised by the upper fluid is partly balanced by a similar Bernoulli effect in the lower fluid, caused by the motion induced by the progressive wave. This motion is proportional to the wave velocity C. The inclusion of only a lower boundary progressively suppresses this motion in the lower fluid and eventually (for sufficiently thin films) leads to the subcritical behavior. When an upper wall is added to the system, the Bernoulli suction is apparently enhanced with a parallel increase in the lower fluid inertia, due to the increase in the phase velocity (see [15] and [16]). However, the allowable wave amplitudes are strongly suppressed because of the small channel height and this provides the essential difference between a gas/liquid and a liquid/liquid system. Because of the strong stratification between a gas and a liquid, small nonlinear corrections in the wave height have a significant stabilizing effect, leading to supercritical behavior. On the contrary, a liquid/liquid system cannot achieve the same degree of gravity stabilization, leading to subcritical behavior. A detailed analysis of all the terms reveals the exact contributions at each order of approximation, but the above qualitative description essentially contains the underlying physics.

The final case considered is that of a shallow upper fluid in contact with a lower fluid of infinite extent. This particular combination could be of interest in the description of systems where the lower fluid is at least an order of magnitude deeper than the upper. The results for the two representative density ratios are plotted in figure 3. For $r = 0.9$ the behavior is the same as for two shallow layers (cf. figure 2c). It is worth noting though that, for $r = 0.1$, the slope S of the dynamical limit curve is a nonmonotonic function of the dimensionless upper depth, exhibiting a local maximum for some intermediate value. The subcritical behavior for very thin upper fluids is in part artificial, emanating from the assumption of a lower fluid of infinite extent. In reality, for sufficiently long waves, the dimensionless liquid depth will be finite and the slope will turn upwards and become positive once again. There is no doubt, however, that for a range of wavelengths, the slope S is negative. The practical significance of this observation is considered in the discussion, after a stability analysis of the above solutions has been performed.

4.2. *The stability of progressive waves of permanent form*

Equation [14], which describes the slow evolution of the wave amplitude under self-interaction, is put in the concise form

$$k\ddot{a} + m(ka) + l(ka)^3 = 0. \quad [18]$$

The progressive waves of section 4.1 correspond to $da/dt = 0$. The stability of these solutions to small perturbations around the steady state, $a = a_0$, is found—by substituting

$$a = a_0 + re^{nt} \quad [19]$$

in [18], linearizing and subtracting the steady solution—to be governed by the relation

$$n^2 + 2lk^2a_0^2 = 0. \quad [20]$$

Therefore, for $S > 0$ ($l > 0$) weakly nonlinear waves are stable, whereas for $S < 0$ ($l < 0$) the waves are unstable. This is a special case of a classical result of nonlinear hydrodynamic stability analysis, termed supercritical stability and subcritical instability, respectively. Therefore, the use of the above terms in the preceding section is now justified. It should be noted that a more general stability analysis by Miles (1986), permitting quasi-periodic solutions as well, slightly modifies the above results in that the subcritical waves becomes unstable only above a certain amplitude.

5. DISCUSSION

The consideration of fluid layers of finite depth uncovers a surprisingly rich behavior of the dynamical limit to weakly nonlinear waves of permanent form. For long waves between thin fluid layers, an increase in the critical current velocity with increasing wave amplitude is typically calculated for gas/liquid systems. This behavior becomes more pronounced as the wavelength increases. When, however, a liquid of infinite extent is considered, there exists a range of wavelengths for which the critical current velocity is a decreasing function of the wave amplitude. For gas spacing of the order of centimeters, this behavior is predicted for waves with length in the order of meters.

Systems of two immiscible liquids of finite depth are shown to exhibit a different behavior. The increase in the critical current velocity with increasing wave amplitude becomes less pronounced and is eventually reversed as the wavelength increases. It should be noted though that such an inviscid analysis is expected to be of practical significance only for liquids with very different viscosities. Otherwise, the velocity profiles would be such that a formulation based on the relative current speed would be unrealistic.

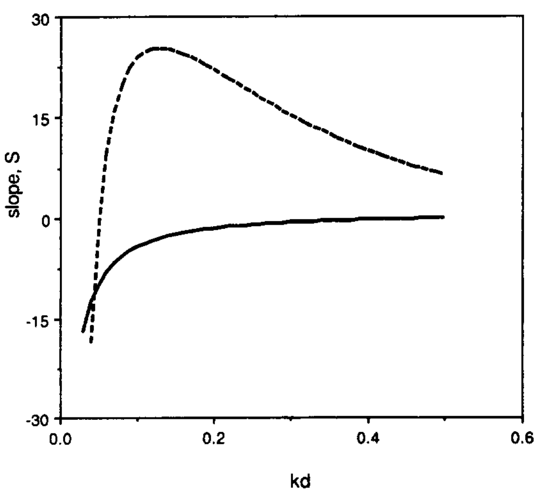

Figure 3. Slope of the dynamical limit line vs the dimensionless depth kd_2 for a lower fluid of infinite extent. Results are presented for the ratio of densities $r = 1.0$ (- - -) and 0.9 (——).

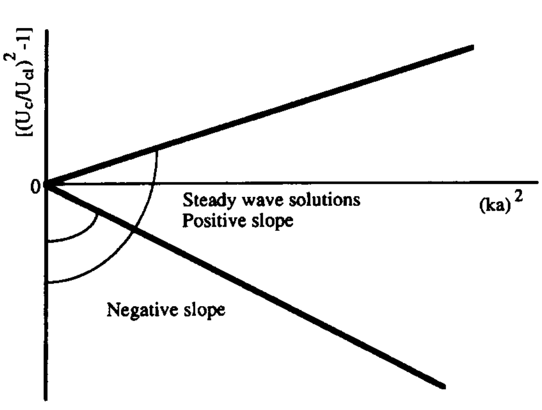

Figure 4. Critical current velocity as a function of wave steepness.

The significance of the behavior of the dynamical limit is demonstrated in figure 4. The term $[(U_c/U_{cl})^2 - 1]$ is plotted vs the wave steepness k^2a^2 for two generic cases, one with positive and one with negative slope. Progressive waves of permanent form exist in the region between the negative y-axis and the dynamical limit line. It is evident that, for a positive slope, the restriction imposed by the dynamical limit is a minimum wave steepness when $U > U_{cl}$. With a negative slope there are no steady solutions for $U > U_{cl}$ and the restriction is a maximum steepness for $U < U_{cl}$. Thus, for the first case, an increase in the current velocity U is expected to lead to progressively higher waves. If interfacial waves generated by a mechanism different from the Kelvin–Helmholtz instability already exist, they are expected to steepen in accordance with the behavior of the steady solution. For a dynamical limit with negative slope, weakly nonlinear theory predicts that the inertia of the fluids and the gravity force cannot balance beyond a certain amplitude for high current velocities. This amplitude tends to zero as the Kelvin–Helmholtz velocity U_{cl} is approached from below.

The stability analysis provides additional information on the evolution of the interface. In the present work the stability of weakly nonlinear waves to self-interaction with higher harmonics is considered. This choice restricts the evolution of the wave to development in time only and not in space and excludes the well known side-band instability (e.g. Drazin & Reid 1981). This instability is known to lead to a periodic modulation and partial recurrence of the original wavetrain (Lake *et al.* 1977). It is sought, therefore, in the present work to decouple this "weak" instability from the possibility of an explosive growth of the wave amplitude, due to a nonlinear Kelvin–Helmholtz mechanism. Evidence for such an evolution has been provided by Ahmed & Banerjee (1985), who noticed a sharp increase of the amplification factor in their stability analysis based on a Schrödinger equation.

The present analysis indicates that weakly nonlinear waves are stable, when the slope of the dynamical limit is positive. This result supports the arguments of the previous paragraph about an orderly growth of the wave steepness with increasing current speed. The results for negative slope are rather puzzling. Waves become unstable at current velocities below U_{cl}, leading to a rapid evolution (all harmonics become important) whose final stage cannot be predicted by the present analysis. For gas/liquid systems such a possibility is supported only for a range of long waves bounded by a thin gas film. Whether this corresponds to the subcritical transition of slug flow, which has been repeatedly documented and proven hard to explain (Wallis & Dobson 1973; Lin & Hanratty 1986), is presently unclear. In contrast to the above conclusions, the analysis of Ahmed & Banerjee (1985) indicates subcritical instability for the entire range of wavelengths in their work. Therefore, it seems plausible that an instability mechanism, different from the nonlinear self-interaction presently considered, is also active. The exact nature of such an instability remains to be elucidated.

Finally, it is noted that a theory has recently been proposed (Bontozoglou & Hanratty 1990) which seems to reconcile to a certain extent the experimental results of transition to slug flow for both water and highly viscous liquids. A period-doubling bifurcation of capillary–gravity ripples is studied therein and it is speculated that this evolution—triggered by a subcritical Kelvin–Helmholtz instability of the short wave—is the first step in a nonlinear process leading from ripples to slugs. The underlying mechanism is essentially a second-order resonance between the short wave and its first subharmonic. With this in mind, the following more general possibility is proposed; a long wave that is itself Kelvin–Helmholtz stable could grow through resonance with a shorter wave that becomes unstable. This possible mechanism will be considered in future work.

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